MA10207 Analysis 1 - Semester 1, 2021/22

Problem Sheet Week 7- Solutions

1. Find $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ for the following sequences

$$a.\quad a_n = (-1)^n + \sin(2\pi n),\quad b.\quad a_n = (-1)^n \frac{2n}{1+n}.$$

Answer. a. Since $\sin(2\pi n) = 0$ for all n, $a_n = (-1)^n$. Since $\sup_{n\geq k}(-1)^n = 1$, then $\limsup_{n\to\infty} a_n = \lim_{k\to\infty}\sup_{n\geq k}(-1)^n = 1$. Since $\inf_{n\geq k}(-1)^n = -1$, then $\liminf_{n\to\infty} a_n = \lim_{k\to\infty}\inf_{n\geq k}(-1)^n = -1$.

b. For any k let $S_k = \{a_n : n \geq k\}$.

Observe that $\lim_{n\to\infty}\frac{2n}{1+n} = 2$.

We claim that $\inf S_k = -2$.

For any $n \geq k$,

$$a_n = (-1)^n \frac{2n}{n+1} \geq -\frac{2n}{n+1} > -2$$

since $\frac{2n}{n+1} < 2$ for all n.

Assume there exists a lower bound t for S_k with $t > -2$. We can take $t < 0$. Then for all $n \geq k$ we have $a_n \geq t$. We assume n is odd. Then

$$a_n \geq t \quad \Leftrightarrow \quad (-1)^n \frac{2n}{n+1} \geq t \quad \Leftrightarrow \quad -\frac{2n}{n+1} \geq t$$

$$-(2+t)n \geq t \quad \Leftrightarrow \quad n \leq -\frac{t}{t+2},$$

where in the latter inequality we used that $-(2+t) < 0$. This is a contradiction with the Archimedean Principle. Thus there is no lower bound to S_k greater than -2. This proves that $\inf_k S_k = -2$. Thus $\liminf_{n\to\infty} a_n = \lim_{k\to\infty}\inf_k S_k = -2$.

We claim that $\sup S_k = 2$.

For any $n \geq k$,

$$a_n = (-1)^n \frac{2n}{n+1} \leq \frac{2n}{n+1} < 2.$$

Assume there exists an upper bound s for S_k with $s < 2$. We can take $s > 0$. Then for all $n \geq k$ we have $a_n \leq s$. We assume n is even. Then

$$a_n \leq s \quad \Leftrightarrow \quad \frac{2n}{n+1} \leq s \quad \Leftrightarrow \quad (2-s)n \leq s \quad \Leftrightarrow \quad n \leq \frac{s}{2-s},$$

where in the latter inequality we used that $2 - s > 0$. This is a contradiction with the Archimedean Principle. Thus there is no upper bound to S_k less than 2. This proves that $\sup_k S_k = 2$. Thus $\limsup_{n\to\infty} a_n = \lim_{k\to\infty}\sup_k S_k = 2$.

2. Let $(a_n)_n$ and $(b_n)_n$ be two sequence. Assume there exists $M \in \mathbb{N}$ such that

$$a_n \leq b_n \quad \forall \quad n \geq M.$$

Prove that

$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n, \quad \text{and} \quad \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n.$$

Answer. For any $k \in \mathbb{N}$, we consider the sets $S_k = \{a_n : n \geq k\}$ and $T_k = \{b_n : n \geq k\}$.

Let $k \geq M$. Then $a_n \leq b_n$ for all $n \geq k$ and

$$\inf_k S_k \leq a_n \leq b_n$$

for all $n \geq k$. Hence

$$\inf_k S_k \leq b_n, \quad \forall n \geq k.$$

Thus $\inf_k S_k \leq \inf_k T_k$ for all $k \geq M$. Henceforth $\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$.

Besides,

$$\sup_k T_k \geq b_n \geq a_n$$

for all $n \geq k$. Observe then that

$$\sup_k T_k \geq a_n, \quad \forall n \geq k.$$

Thus $\sup_k T_k \geq \sup_k S_k$ for all $k \geq M$. Henceforth $\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n$.

3. Compute

$$a. \quad \sum_{n=1}^{\infty} \frac{1}{4n^2-1}, \quad b. \quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}.$$

Answer. a. Let $n \in \mathbb{N}$ and consider

$$S_n = \sum_{k=1}^{n} \frac{1}{4k^2-1}.$$

Since

$$\frac{1}{4k^2-1} = \frac{1}{2}[\frac{1}{2k-1} - \frac{1}{2k+1}]$$

we get

$$\begin{aligned}
S_n &= \frac{1}{2}\sum_{k=1}^{n}[\frac{1}{2k-1} - \frac{1}{2k+1}] \\
&= \frac{1}{2}[1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + ..\frac{1}{2n-1} - \frac{1}{2n+1}] \\
&= \frac{1}{2}[1 - \frac{1}{2n+1}].
\end{aligned}$$

Hence, by the Algebra of Limits,

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{2}[1 - \frac{1}{2n+1}]\right) = \frac{1}{2}.$$

b. Let $n \in \mathbb{N}$ and consider

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)}.$$

Since

$$\frac{1}{k(k+1)(k+2)} = \frac{A}{k(k+1)} + \frac{B}{(k+1)(k+2)} \quad \Leftrightarrow \quad A = \frac{1}{2},\ B = -\frac{1}{2},$$

we have

$$\begin{aligned}
S_n &= \sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} \\
&= \frac{1}{2}\sum_{k=1}^{n}[\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)}] \\
&= \frac{1}{2}[\frac{1}{2} - \frac{1}{6} + \frac{1}{6} - \frac{1}{12} + \dots + \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}] \\
&= \frac{1}{2}[\frac{1}{2} - \frac{1}{(n+1)(n+2)}].
\end{aligned}$$

We then conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{1}{2}[\frac{1}{2} - \frac{1}{(n+1)(n+2)}]\right] = \frac{1}{4}.$$

4. a. Prove that if $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \frac{a_n}{n^2}$ converges.

   b. Prove that if $\lim_{n\to\infty} a_n = 1$, then $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}}$ diverges.

Answer. a. Let $S_n = \sum_{k=1}^{n} \frac{a_k}{k^2}$ be the sequence of partial sums associated to the series $\sum_{n=1}^{\infty} \frac{a_n}{n^2}$. If we prove that the sequence $(S_n)_n$ is convergent, then $\sum_{n=1}^{\infty} \frac{a_n}{n^2}$ converges.

We prove that $(S_n)_n$ converges showing that it is a Cauchy sequence.

Let $\varepsilon > 0$. For any $m, n \in \mathbb{N}$, $m > n$

$$|S_m - S_n| = |\sum_{k=n+1}^{m} \frac{a_k}{k^2}| = \sum_{k=n+1}^{m} \frac{a_k}{k^2} \leq \sum_{k=n+1}^{m} a_k.$$

Since $\sum_{n=1}^{\infty} a_n$ converges, the sequence $T_n = \sum_{k=1}^{n} a_k$ converges, hence it is a Cauchy sequence. Thus there exists N such that, for all $m, n > N$, $m > n$,

$$|S_m - S_n| \leq |T_m - T_n| \leq \sum_{k=n+1}^{m} a_k \leq \varepsilon.$$

This proves that $(S_n)_n$ is a Cauchy sequence, hence it converges.

b. Since $\lim_{n\to\infty} a_n = 1$, there exists $N \in \mathbb{N}$ such that

$$|a_n - 1| < \frac{1}{2} \quad \forall \quad n \geq N \quad \Leftrightarrow \quad \frac{1}{2} < a_n < \frac{3}{2} \quad \forall \quad n \geq N.$$

Let $k > N$. Then

$$\begin{aligned} \sum_{n=1}^{k} \frac{a_n}{\sqrt{n}} &= \sum_{n=1}^{N} \frac{a_n}{\sqrt{n}} + \sum_{n=N+1}^{k} \frac{a_n}{\sqrt{n}} \\ &\geq \sum_{n=1}^{N} \frac{a_n}{\sqrt{n}} + \frac{1}{2} \sum_{n=N+1}^{k} \frac{1}{\sqrt{n}} \end{aligned}$$

Observe that

$$\sum_{n=N+1}^{k} \frac{1}{\sqrt{n}} \geq \frac{k-N}{\sqrt{k+1}}.$$

Thus

$$\sum_{n=1}^{k} \frac{a_n}{\sqrt{n}} \geq \sum_{n=1}^{N} \frac{a_n}{\sqrt{n}} + \frac{1}{2}\frac{k-N}{\sqrt{k+1}}$$

and

$$\infty = \lim_{k\to\infty} \left[\sum_{n=1}^{N} \frac{a_n}{\sqrt{n}} + \frac{1}{2}\frac{k-N}{\sqrt{k+1}} \right] \leq \lim_{k\to\infty} \sum_{n=1}^{k} \frac{a_n}{\sqrt{n}}.$$

Thus the series $\sum_{n=1}^{k} \frac{a_n}{\sqrt{n}}$ diverges.

Homework.

1. Let $x_n = 0$ if n is odd, and $x_n = 1 - \frac{1}{n}$ if n is even.

   Evaluate $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$.

Answer. For all $k \in \mathbb{N}$ let $S_k = \{x_n \mid n \geq k\}$; then 0 is a lower bound for S_k and $0 \in S_k$ so $\inf S_k = 0$. Thus $\liminf_{n\to\infty} x_n = 0$.

For all $k \in \mathbb{N}$ we have $x_k < 1$ so 1 is an upper bound for S_k.

If $t < 1$ we can choose $N \in \mathbb{N}$ such that $1/N < 1 - t$ (by the Archimedean Postulate);

if $k \in \mathbb{N}$ we can choose an even integer $m \geq \max\{N, k\}$, then $x_m > t$ and $x_m \in S_k$ so t is not an upper bound for S_k.

Thus $\sup S_k = 1$ for every $k \in \mathbb{N}$, so $\limsup_{n\to\infty} x_n = 1$.

2. Compute

$$a. \quad \sum_{n=1}^{\infty} \frac{1}{n(n+3)}, \qquad b. \quad \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}.$$

Answer. a. We have

$$\frac{1}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3} \quad \Leftrightarrow \quad A = \frac{1}{3}, \quad B = -\frac{1}{3}.$$

Hence

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k(k+3)} &= \frac{1}{3}\sum_{k=1}^{n}[\frac{1}{k} - \frac{1}{k+3}] \\
&= \frac{1}{3}(1 - \frac{1}{4} + \frac{1}{2} - \frac{1}{5} + \frac{1}{3} - \frac{1}{6} + \frac{1}{4} - \frac{1}{7} + ... + \frac{1}{k} - \frac{1}{n+3}) \\
&= \frac{1}{3}(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+3}) = \frac{1}{3}(\frac{11}{6} - \frac{1}{n+3}).
\end{aligned}$$

So we have

$$\sum_{k=1}^{\infty} \frac{1}{k(k+3)} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k(k+3)} = \lim_{n\to\infty} \frac{1}{3}(\frac{11}{6} - \frac{1}{n+3}) = \frac{11}{18}.$$

b. Observe that $2n+1 = (n+1)^2 - n^2$. Hence

$$\begin{aligned}
\sum_{n=1}^{k} \frac{2n+1}{n^2(n+1)^2} &= \sum_{n=1}^{k} \frac{(n+1)^2 - n^2}{n^2(n+1)^2} \\
&= \sum_{n=1}^{k}[\frac{1}{n^2} - \frac{1}{(n+1)^2}] = 1 - \frac{1}{(k+1)^2}
\end{aligned}$$

and

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \lim_{k\to\infty} \sum_{n=1}^{k} \frac{2n+1}{n^2(n+1)^2} = \lim_{k\to\infty}\left(1 - \frac{1}{(k+1)^2}\right) = 1.$$

3. When $|a| < 1$, by calculating the partial sums, evaluate $\sum_{n=1}^{\infty} na^n$.

[Hint: Prove first that $\sum_{n=1}^{N} na^n = \frac{a}{1-a}\left[\frac{1-a^N}{1-a} - Na^N\right]$.]

Answer. We prove by induction that

$$\sum_{n=1}^{N} na^n = \frac{a}{1-a}\left[\frac{1-a^N}{1-a} - Na^N\right]. \qquad (*)$$

Let $\Lambda = \{N \in \mathbb{N} : (*) \quad \text{holds true}\}$. We have that $1 \in \Lambda$ since

$$a = \frac{a}{1-a}[\frac{1-a}{1-a} - a].$$

If $N \in \Lambda$, then

$$\begin{aligned}
\sum_{n=1}^{N+1} na^n &= \sum_{n=1}^{N} na^n + (N+1)a^{N+1} \\
&= \frac{a}{1-a}\left[\frac{1-a^N}{1-a} - Na^N\right] + (N+1)a^{N+1} \\
&= \frac{a}{1-a}\left[\frac{1-a^N}{1-a} - Na^N + (N+1)a^N(1-a)\right] \\
&= \frac{a}{1-a}\left[\frac{1-a^N}{1-a} - Na^N + (N+1)a^N - (N+1)a^{N+1}\right] \\
&= \frac{a}{1-a}\left[\frac{1-a^N}{1-a} + a^N - (N+1)a^{N+1}\right] \\
&= \frac{a}{1-a}\left[\frac{1-a^N + (1-a)a^N}{1-a} - (N+1)a^{N+1}\right] \\
&= \frac{a}{1-a}\left[\frac{1-a^{N+1}}{1-a} - (N+1)a^{N+1}\right].
\end{aligned}$$

This implies that $N+1 \in \Lambda$. By induction, formula $(*)$ is valid for any integer N.

Since $|a| < 1$, by Corollary 58 in the Lecture Notes $\lim_{N\to\infty} Na^N = 0$. Using the Algebra of Limits, one has

$$\begin{aligned}
\sum_{n=1}^{\infty} na^n &= \lim_{N\to\infty} \sum_{n=1}^{N} na^n \\
&= \lim_{N\to\infty} \frac{a}{1-a}\left[\frac{1-a^N}{1-a} - Na^N\right] = \frac{a}{(1-a)^2}.
\end{aligned}$$

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