MA10207 Analysis 1 - Semester 1, 2021/22

Problem Sheet Week 6 - Solutions

1. By looking for suitable subsequences, prove that the following sequences are not convergent:

$$a.\quad \left(\frac{(-1)^n n}{2n+1}\right)_n, \qquad b.\quad \left(\sin(\frac{n\pi}{3})\right)_n.$$

Answer. Corollary 62 from the lecture notes states

If a sequence has two subsequences converging to different limits, then the sequence is divergent (or not convergent).

a. Let $a_n = \frac{(-1)^n n}{2n+1}$. Take the subsequences $(a_{2k})_k$ and $(a_{2k-1})_k$. Then $a_{2k} = \frac{2k}{4k+1}$ and thus $\lim_{k\to\infty} a_{2k} = \frac{1}{2}$. Besides, $a_{2k-1} = \frac{-(2k-1)}{4k-1}$ and thus $\lim_{k\to\infty} a_{2k-1} = -\frac{1}{2}$. We conclude that the sequence $(\frac{(-1)^n n}{2n+1})_n$ is not convergent.

b. Let $b_n = \sin(\frac{n\pi}{3})$. Take the subsequences $(b_{6k})_k$ and $(b_{1+6k})_k$. Then $b_{6k} = \sin(\frac{6k\pi}{3}) = \sin(2k\pi) = 0$ and thus $\lim_{k\to\infty} a_{6k} = 0$. Besides, $b_{1+6k} = \sin(\frac{(1+6k)\pi}{3}) = \sin(\frac{\pi}{3} + 2k\pi) = \frac{\sqrt{3}}{2}$ and thus $\lim_{k\to\infty} b_{1+6k} = \frac{\sqrt{3}}{2}$. We conclude that the sequence $\sin(\frac{n\pi}{3})$ is not convergent.

2. Let (a_n) be a sequence such that the subsequences (a_{2k}) and (a_{2k-1}) both converge to the same limit L. Prove that $\lim_{n\to\infty} a_n = L$.

Answer. We want to prove that, for any $\varepsilon > 0$ there exists N such that

$$|a_n - L| < \varepsilon,$$

for all $n \geq N$.

Since $\lim_{k\to\infty} a_{2k} = L$ and $\lim_{k\to\infty} a_{2k-1} = L$, given $\varepsilon > 0$ there exist K_1 and K_2 such that

$$|a_{2k} - L| < \varepsilon, \quad \text{for all} \quad k \geq K_1, \tag{1}$$

and

$$|a_{2k-1} - L| < \varepsilon, \quad \text{for all} \quad k \geq K_2. \tag{2}$$

Set $N = \max(2K_1, 2K_2 - 1)$. If $n \geq N$ and n is even, then $n = 2\ell \geq N \geq 2K_1$, thus $\ell \geq K_1$ and from (1) we get $|a_n - L| < \varepsilon$. If $n \geq N$ and n is odd, then $n = 2\ell - 1 \geq$

$N \geq 2K_2 - 1$, thus $\ell \geq K_2$ and from (2) we get $|a_n - L| < \varepsilon$. Thus for all $n \geq N$, one has $|a_n - L| < \varepsilon$. We conclude that $\lim_{n\to\infty} a_n = L$.

3. Compute the limits of the following sequences

$$a. \quad a_n = \frac{2^n}{1 + 3 \cdot 2^n}, \quad b. \quad a_n = \left(\frac{n^2 - 3}{n^2}\right)^n, \quad c. \quad a_n = \frac{7^n + n^3 4^n}{n^{10} - 7^n}.$$

[*Hint: use the binomial inequality for b.*]

Answer. a. We claim that $2^n > n$, for any $n \in \mathbb{N}$. We prove it by induction. For $n = 1$, we observe that $2 > 1$. Assume that $2^n > n$, then

$$2^{n+1} - n - 1 = 2^n - n + 2^n - 1 > 0.$$

Since $\lim_{n\to\infty} n = \infty$, it holds that $\lim_{n\to\infty} 2^n = \infty$, and $\lim_{n\to\infty} \frac{1}{2^n} = 0$. Thus

$$\lim_{n\to\infty} \frac{2^n}{1 + 3\,2^n} = \lim_{n\to\infty} \frac{1}{3 + \frac{1}{2^n}} = \frac{1}{3}.$$

b. We write $a_n = \left(1 - \frac{3}{n^2}\right)^n$, which gives $a_n \leq 1$ for any n. The binomial inequality gives

$$\left(1 - \frac{3}{n^2}\right)^n \geq 1 - \frac{3}{n^2} n.$$

Thus we have

$$1 - \frac{3}{n} \leq \left(1 - \frac{3}{n^2}\right)^n < 1.$$

The Pinching Theorem implies that $\lim_{n\to\infty} a_n = 1$, since $\lim_{n\to\infty}(1 - \frac{3}{n}) = 1$.

c. It holds

$$\lim_{n\to\infty} \frac{7^n + n^3 4^n}{n^{10} - 7^n} = -\lim_{n\to\infty} \frac{1 + n^3(\frac{4}{7})^n}{1 - n^{10}(\frac{1}{7})^n} = -1,$$

since by Corollary 57 in the lecture notes we have

$$\lim_{n\to\infty} n^3(\frac{4}{7})^n = 0, \quad \lim_{n\to\infty} n^{10}(\frac{1}{7})^n = 0$$

4. The sequence $(a_n)_n$ satisfies $a_n > 0$ and $a_{n+1} < \frac{a_n}{2}$ for all n. Prove that $\lim_{n\to\infty} a_n = 0$.

Answer. We claim that $a_{n+1} < \frac{a_1}{2^n}$ for all n. We prove it by induction. For $n = 1$, we have that $a_2 < \frac{a_1}{2}$ by assumption. Assume that $a_{n+1} < \frac{a_1}{2^n}$. Then we have

$$a_{n+2} < \frac{a_{n+1}}{2} < \frac{1}{2}\frac{a_1}{2^n} = \frac{a_1}{2^{n+1}}.$$

Thus, we have that $a_{n+1} < \frac{a_1}{2^n}$ for all n.

For all n, we have

$$0 < a_n < \frac{a_1}{2^{n+1}}.$$

Since $\lim_{n\to\infty} \frac{a_1}{2^{n+1}} = 0$, by the Sandwich Theorem we conclude that

$$\lim_{n\to\infty} a_n = 0.$$

Homework.

1. Find the following limits.

$$a.\quad \lim_{n\to\infty} \frac{n^n}{n!} \qquad b.\quad \lim_{n\to\infty} \frac{3^n}{n^2},$$

$$c.\quad \lim_{n\to\infty} n^2 \left(\frac{2}{3}\right)^n, \qquad d.\quad \lim_{n\to\infty} \frac{n^3+2n^2+1}{n^4+8n^2+2},$$

Answer. a. Use the growth factor test, and compute

$$\begin{aligned}
\lim_{n\to\infty} \frac{(n+1)^{(n+1)}}{(n+1)!}\frac{n!}{n^n} &= \lim_{n\to\infty} \frac{(n+1)^{(n+1)}}{n^n}\frac{n!}{(n+1)!} \\
&= \lim_{n\to\infty} \frac{(n+1)^n(n+1)}{n^n}\frac{1}{(n+1)} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n \\
&= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n.
\end{aligned}$$

By Example 50 from the lecture notes, $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n > 1$, thus $\lim_{n\to\infty} \frac{n^n}{n!} = \infty$.

b. Use the growth factor test, and compute

$$\lim_{n\to\infty} \frac{3^{n+1}}{(n+1)^2}\frac{n^2}{3^n} = \lim_{n\to\infty} 3\left(1-\frac{1}{n+1}\right)^2 = 3 > 1,$$

thus $\lim_{n\to\infty} \frac{3^n}{n^2} = \infty$.

c. Use the growth factor test, and compute

$$\lim_{n\to\infty} (n+1)^2 \left(\frac{2}{3}\right)^{n+1} \frac{1}{n^2\left(\frac{2}{3}\right)^n} = \lim_{n\to\infty} \frac{2}{3}(1+\frac{1}{n})^2 = \frac{2}{3} < 1,$$

thus $\lim_{n\to\infty} n^2 \left(\frac{2}{3}\right)^n = 0$.

d. By the algebra of limits

$$\lim_{n\to\infty} \frac{n^3+2n^2+1}{n^4+8n^2+2} = \lim_{n\to\infty} \frac{1}{n}\frac{1+2\frac{1}{n}+\frac{1}{n^3}}{1+8\frac{1}{n^2}+\frac{2}{n^4}} = 0$$

since $\lim_{n\to\infty} \frac{1}{n} = 0$.

2. Prove that (a_n) is convergent if and only if the subsequences $(a_{2k})_k$, $(a_{2k-1})_k$ and $(a_{3k})_k$ converge.

Answer. Assume first that (a_n) is convergent. By Proposition 61 in the Lecture Notes it follows that any subsequence of (a_n) is convergent. Thus $(a_{2k})_k$, $(a_{2k-1})_k$ and $(a_{3k})_k$ converge.

Assume now that $(a_{2k})_k$, $(a_{2k-1})_k$ and $(a_{3k})_k$ converge. We want to show that (a_n) is convergent.

Let

$$L_1 = \lim_{k\to\infty} a_{2k}, \quad L_2 = \lim_{k\to\infty} a_{2k-1}, \quad L_3 = \lim_{k\to\infty} a_{3k}.$$

We claim that $L_1 = L_3$. Let $\varepsilon > 0$. Since $(a_{3k})_k$ is convergent, it is a Cauchy sequence. Thus

$$\exists K_1 \quad \forall h, j \geq K_1 \quad |a_{3h} - a_{3j}| < \frac{\varepsilon}{3}.$$

Since $(a_{2k})_k$ is convergent, its subsequence $(a_{6m})_m$ is also convergent and $\lim_{m\to\infty} a_{6m} = L_1$. Hence

$$\exists M \quad \forall m \geq M \quad |a_{6m} - L_1| < \frac{\varepsilon}{3}.$$

Let $K = \max(K_1, M)$. Then, for all $k \geq K$

$$|a_{3k} - L_1| = |a_{3k} - a_{6k}| + |a_{6k} - L_1| < \varepsilon.$$

We proved that $\lim_{k\to\infty} a_{3k} = L_1$. Since the limit, when it exists, it is unique, we get $L_1 = L_3$.

We claim that $L_2 = L_3$. Let $\varepsilon > 0$. Since $(a_{3k})_k$ is convergent, it is a Cauchy sequence. Thus

$$\exists K_1 \quad \forall h, j \geq K_1 \quad |a_{3h} - a_{3j}| < \frac{\varepsilon}{3}.$$

Since $(a_{2k-1})_k$ is convergent, its subsequence $(a_{2(3m-1)-1})_m = (a_{3(2m-1)})_m$ is also convergent and $\lim_{m\to\infty} a_{3(2m-1)} = L_2$. Hence

$$\exists M \quad \forall m \geq M \quad |a_{3(2m-1)} - L_2| < \frac{\varepsilon}{3}.$$

Let $K = \max(K_1, M)$. Then, for all $k \geq K$

$$|a_{3k} - L_1| = |a_{3k} - a_{3(2k-1)}| + |a_{3(2k-1)} - L_2| < \varepsilon.$$

Since the limit, when it exists, it is unique, we get $L_2 = L_3$.

Since $L_1 = L_2 = L_3$, we have that $\lim_{k\to\infty} a_{2k} = \lim_{k\to\infty} a_{2k-1} = L_1$.

We have proven in Execise 2 the following:

Let (a_n) be a sequence such that the subsequences (a_{2k}) and (a_{2k-1}) both converge to the same limit L. Prove that $\lim_{n\to\infty} a_n = L$.

Hence $\lim_{n\to\infty} a_n$ exists.

3. a. Prove that a monotone sequence which contains a bounded subsequence is bounded.

   b. Prove that $(a_n)_{n\in\mathbb{N}}$ is not bounded below if and only if there exists a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ such that $\lim_{k\to\infty} a_{n_k} = -\infty$.

Answer. a. Let $(a_n)_n$ be a monotone sequence and let $(a_{n_k})_k$ be a bounded subsequence of $(a_n)_n$.

Since $(a_{n_k})_k$ is bounded, there exists $M > 0$ such that

$$|a_{n_k}| \le M \quad \forall k \in \mathbb{N} \quad \Leftrightarrow \quad -M \le a_{n_k} \le M \quad \forall k \in \mathbb{N}.$$

If $(a_n)_n$ is monotone *increasing*, then we need to prove that $(a_n)_n$ is bounded above. Since $k \to n_k$ is strictly increasing, then $n_k \ge k$ for all $k \in \mathbb{N}$. (You can prove this fact by induction.) Since $(a_k)_k$ is monotone *increasing*, $a_k \le a_{n_k} \le M$. Thus $(a_k)_k$ is bounded.

If $(a_n)_n$ is monotone *decreasing*, then we need to prove that $(a_n)_n$ is bounded below. Since $k \to n_k$ is strictly increasing, then $n_k \ge k$ for all $k \in \mathbb{N}$. Since $(a_k)_k$ is monotone *decreasing*, $a_k \ge a_{n_k} \ge -M$. Thus $(a_n)_n$ is bounded.

b. Assume $(a_n)_n$ is not bounded below. Then there exists $n_1 \in \mathbb{N}$ so that $a_{n_1} < -1$. Since $(a_n)_n$ is not bounded below, we now can find n_2, with $n_2 > n_1$ so that

$$a_{n_2} < -2.$$

Repeating this, we can prove that for all $k \in \mathbb{N}$, there exists $n_k > n_{k-1}$

$$(\forall k \in \mathbb{N}) \quad (\exists n_k \in \mathbb{N}) \quad a_{n_k} \le -k.$$

Thus $(a_{n_k})_k$ is a subsequence of $(a_n)_n$ and $\lim_{k\to\infty} a_{n_k} = -\infty$.

Assume now there exits a subsequence $(a_{n_k})_k$ of $(a_n)_n$ with $\lim_{k\to\infty} a_{n_k} = -\infty$. Hence, for all M there exists $k \in \mathbb{N}$ such that $a_{n_k} < M$. Henceforth $(a_n)_n$ is not bounded below.

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