MA10207 Analysis 1 - Semester 1, 2021/22

Problem Sheet Week 5 - Solutions

1. (*i*) Let $a \in (0,1)$. Show that $\lim_{n\to\infty}(1+a^n)^{\frac{1}{n}} = 1$.

   (*ii*) Let $c > d > 0$. Show that $\lim_{n\to\infty}(c^n + d^n)^{\frac{1}{n}} = c$

   (*iii*) Compute $\lim_{n\to\infty}(3^{2n} + n^{17}3^n)^{\frac{1}{n}}$.

   Answer. (*i*) We write $(1+a^n)^{\frac{1}{n}} = 1 + x_n$ for some $x_n > 0$. Then

$$1 + a^n = (1+x_n)^n \geq 1 + nx_n$$

by the binomial inequality. Hence

$$0 \leq nx_n \leq a^n \quad \Rightarrow \quad 0 \leq x_n \leq \frac{a^n}{n}.$$

Since $a \in (0,1)$, $\lim_{n\to\infty} \frac{a^n}{n} = 0$, and by comparison also $\lim_{n\to\infty} x_n = 0$. We conclude that

$$\lim_{n\to\infty}(1+a^n)^{\frac{1}{n}} = \lim_{n\to\infty}(1+x_n) = 1.$$

(*ii*) Since $c > d > 0$, we have that $0 < \frac{d}{c} < 1$ and hence, from (*i*), $\lim_{n\to\infty}(1 + (\frac{d}{c})^n)^{\frac{1}{n}} = 1$. Thus, by the Algebra of Limits,

$$\lim_{n\to\infty}(c^n + d^n)^{\frac{1}{n}} = \lim_{n\to\infty} c\left(1 + (\frac{d}{c})^n\right)^{\frac{1}{n}} = c,$$

(*iii*) We write

$$\lim_{n\to\infty}(3^{2n} + n^{17}3^n)^{\frac{1}{n}} = \lim_{n\to\infty} 9(1 + n^{17}3^{-n})^{\frac{1}{n}}$$

We write $(1 + n^{17}3^{-n})^{\frac{1}{n}} = 1 + x_n$ for some $x_n > 0$. Then

$$1 + n^{17}3^{-n} = (1+x_n)^n \geq 1 + nx_n$$

by the binomial inequality. Hence

$$0 \leq nx_n \leq n^{17}3^{-n} \quad \Rightarrow \quad 0 \leq x_n \leq n^{16}3^{-n}.$$

Since $\lim_{n\to\infty} n^{16}3^{-n} = 0$, by the Sandwich Theorem we have that $\lim_{n\to\infty} x_n = 0$ and

$$\lim_{n\to\infty}(1 + n^{17}3^{-n})^{\frac{1}{n}} = \lim_{n\to\infty}(1+x_n) = 1.$$

By the Algebra of Limits, we conclude that

$$\lim_{n\to\infty}(3^{2n} + n^{17}3^n)^{\frac{1}{n}} = \lim_{n\to\infty} 9(1 + n^{17}3^{-n})^{\frac{1}{n}} = 9.$$

2. Determine if the following sequences

$$(a_n)_{n\in\mathbb{N}} = \left(\frac{n+1}{n+10}\right)_{n\in\mathbb{N}}, \qquad (b_n)_{n\in\mathbb{N}} = \left(\frac{5^{n+1}}{2^n 3^n}\right)_{n\in\mathbb{N}}.$$

are increasing or decreasing. Find their limits, if they exist.

Answer. Since

$$a_{n+1} - a_n = \frac{9}{(n+11)(n+10)} > 0, \quad \forall n \in \mathbb{N}$$

the sequence is increasing. Since $a_n = 1 - \frac{9}{n+10} < 1$, the sequence is convergent. By the Algebra of Limits we have

$$\lim_{n\to\infty} \frac{n+1}{n+10} = \lim_{n\to\infty} 1 - \frac{9}{n+10} = 1,$$

since $\lim_{n\to\infty} \frac{9}{n+10} = 0$.

Since

$$\frac{b_{n+1}}{b_n} = \frac{5}{6} < 1, \quad \forall n \in \mathbb{N}$$

the sequence is decreasing. Since the sequence is bounded below by 0, the sequence is converging. We have

$$\lim_{n\to\infty} \frac{5^{n+1}}{2^n 3^n} = \lim_{n\to infty} \frac{5^{n+1}}{6^n} = \lim_{n\to\infty} 5\left(\frac{5}{6}\right)^n = 0,$$

since $\lim_{n\to\infty} \left(\frac{5}{6}\right)^n = 0$.

3. Show that the sequence

$$a_1 = \frac{1}{4}, \quad a_{n+1} = \frac{a_n}{2} + a_n^2, \quad \forall n \geq 1$$

is convergent.

Answer. The first terms of the sequence are $\frac{1}{4}, \frac{3}{16}, \frac{33}{256}, ..$ All terms in the sequence are positive, thus the sequence is bounded below by 0.

By induction we prove that $a_n < \frac{1}{2}$ for all n. If $n = 1$, $a_1 < \frac{1}{2}$. Suppose $a_n < \frac{1}{2}$. Then

$$\frac{1}{2} - a_{n+1} = \frac{1}{2} - \frac{a_n}{2} - a_n^2 > \frac{1}{2} - \frac{1}{4} - \frac{1}{4} = 0$$

We conclude that $a_n < \frac{1}{2}$ for all n.

Next we prove that $(a_n)_{n\in\mathbb{N}}$ is decreasing. We have

$$a_{n+1} < a_n \Longleftrightarrow a_n^2 - \frac{a_n}{2} < 0 \Longleftrightarrow 0 < a_n < \frac{1}{2}.$$

We conclude that $(a_n)_{n\in\mathbb{N}}$ is decreasing.

The sequence $(a_n)_{n\in\mathbb{N}}$ is monotone decreasing and bounded. Hence the sequence has a limit, $\lim_{n\to\infty} a_n = a$. By the algebra of limits, we have

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{a_n}{2} + a_n^2 = \frac{a}{2} + a^2,$$

thus $a^2 - \frac{a}{2} = 0$. The solutions to this equation are $a = 0$ and $a = \frac{1}{2}$. Since $(a_n)_{n\in\mathbb{N}}$ is decreasing, we have $a_n \leq a_1 = \frac{1}{4}$ for all $n \in \mathbb{N}$. Thus also $a \leq \frac{1}{4}$. Henceforth, $a = 0$.

4. Which of the following statements are true and which are false? Justify your assertions.
   a. If $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = L > 0$, then $\lim_{n\to\infty} a_n b_n = \infty$.
   b. If $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = -\infty$, then $\lim_{n\to\infty}(a_n + b_n) = 0$.
   c. If $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = -\infty$, then $\lim_{n\to\infty} a_n b_n = -\infty$.

Answer. a. True. We want to show that for any M there exists N so that for all $n \geq N$

$$a_n b_n > M.$$

Since $\lim_{n\to\infty} b_n = L$, there exists N_1 so that for all $n \geq N_1$

$$|b_n - L| < \frac{L}{2} \quad \Leftrightarrow \quad \frac{L}{2} < b_n < \frac{3L}{2}.$$

Since $\lim_{n\to\infty} a_n = \infty$, there exists N_2 so that for al $n \geq N_2$

$$a_n > \frac{2M}{L}.$$

Take $N = \max(N_1, N_2)$. Then, for all $n \geq N$,

$$a_n b_n > a_n \frac{L}{2} > \frac{2M}{L}\frac{L}{2} = M.$$

We conclude that $\lim_{n\to\infty} a_n b_n = \infty$.

b. False. Take $a_n = n^2$ and $b_n = -n$. Then $\lim_{n\to\infty} a_n = \infty$, $\lim_{n\to\infty} b_n = -\infty$ and

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty}(n^2 - n) = \lim_{n\to\infty} n^2(1 - \frac{1}{n}) = \infty.$$

c. True. We want to show that for any M there exists N so that for all $n \geq N$

$$a_n b_n < M.$$

It is not restrictive to assume that $M < 0$. Since $\lim_{n\to\infty} a_n = \infty$, there exists N_1 so that for al $n \geq N_1$

$$a_n > \sqrt{|M|}.$$

Since $\lim_{n\to\infty} b_n = -\infty$, there exists N_2 so that for al $n \geq N_1$

$$b_n < -\sqrt{|M|}.$$

Take $N = \max(N_1, N_2)$. Then, for all $n \geq N$,

$$a_n b_n < -\sqrt{|M|}\sqrt{|M|} = M.$$

We conclude that $\lim_{n\to\infty} a_n b_n = -\infty$.

Homework.

1. Determine whether the following sequence

$$a_n = \left(\frac{n}{n^2+1}\right)_{n\in\mathbb{N}}$$

is increasing or decreasing. Find its limit, if it exists.

Answer. Since

$$a_{n+1} - a_n = -\frac{n^2+n-1}{(n^2+1)[(n+1)^2+1]} < 0, \quad \forall n,$$

the sequence is decreasing. Since it is bounded below by 0, the sequence is convergent. By the algebra of limits, we obtain that

$$\lim_{n\to\infty} \frac{n}{n_2+1} = \lim_{n\to\infty} \frac{1}{n}\frac{1}{1+\frac{1}{n^2}} = 0$$

since $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1$.

2. Let $(a_n)_n$ be a real sequence such that $\lim_{n\to\infty} a_n = \infty$. Prove that $\lim_{n\to\infty} \frac{a_n^2+1}{2a_n^2+7} = \frac{1}{2}$.

Answer. We want to prove that

$$(\forall\varepsilon > 0) \quad (\exists N \in \mathbb{N}) \quad \forall n \geq N \quad |\frac{a_n^2+1}{2a_n^2+7} - \frac{1}{2}| < \varepsilon.$$

Let $\varepsilon > 0$. Observe that

$$|\frac{a_n^2+1}{2a_n^2+7} - \frac{1}{2}| < \varepsilon \quad \Leftrightarrow \quad |\frac{-5}{4a_n^2+14}| < \varepsilon \quad \Leftrightarrow \quad \frac{5}{4a_n^2+14} < \varepsilon \quad \Leftrightarrow$$

$$4a_n^2 + 14 > 5\varepsilon^{-1} \quad \Leftrightarrow \quad a_n^2 > \frac{5-14\varepsilon}{4\varepsilon}.$$

This latter inequality is authomatically satisfied if $5 - 14\varepsilon < 0$. Let us assume that $5 - 14\varepsilon \geq 0$. Since $\lim_{n\to\infty} a_n = \infty$, there exists $N \in \mathbb{N}$ such that

$$a_n > \sqrt{\frac{5-14\varepsilon}{4\varepsilon}} \quad \Rightarrow \quad a_n^2 > \frac{5-14\varepsilon}{4\varepsilon}.$$

This concludes the proof.

3. Let $A \subseteq \mathbb{R}$ be a set bounded below, and $A \neq \emptyset$. Show that there exists a sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n \in A$ for any $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \inf(A)$.

Answer. Since A is bounded below, and $A \neq \emptyset$, $\inf(A)$ exists by the Completeness Axiom. By definition of $\inf(A)$, for any $n \in \mathbb{N}$ there exists $a_n \in A$ so that

$$a_n \leq \inf(A) + \frac{1}{n}.$$

Moreover, for all $n \in \mathbb{N}$, $a_n \geq \inf(A)$. Hence, for all n

$$\inf(A) \leq a_n \leq \inf(A) + \frac{1}{n}.$$

We claim that $\lim_{n\to\infty} a_n = \inf(A)$. Observe that

$$0 \leq a_n - \inf(A) \leq \frac{1}{n}.$$

By the Archimedean Postulate, for any ε there exists $N \in \mathbb{N}$ so that $N > \frac{1}{\varepsilon}$. Therefore, for any $n \geq N$,

$$|a_n - \inf(A)| < \varepsilon.$$

Hence $\lim_{n\to\infty} a_n = \inf(A)$.

4. Show that the sequence

$$a_1 = 1, \quad a_{n+1} = -1 + \frac{a_n}{2}, \quad \forall n \geq 1$$

is convergent.

Answer. The first terms of the sequence are $1, -\frac{1}{2}, -\frac{5}{4}, \dots$. First we claim that $a_n \geq -2$ for any n. We prove it by induction. Observe that $a_1 \geq -2$. Assume now that $a_n \geq -2$ and compute

$$a_{n+1} + 2 = -1 + \frac{a_n}{2} + 2 = 1 + \frac{a_n}{2} \geq 1 - 1 = 0.$$

We conclude that $a_n \geq -2$ for any n.

Second we claim that $(a_n)_{n\in\mathbb{N}}$ is decreasing:

$$a_{n+1} - a_n = -1 + \frac{a_n}{2} - a_n = -1 - \frac{a_n}{2} \leq 0, \quad \forall n,$$

because $a_n \geq -2$.

Since $(a_n)_{n\in\mathbb{N}}$ is decreasing and bounded below, it is convergent, $\lim_{n\to\infty} a_n = a$. By the algebra of limits

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left(-1 + \frac{a_n}{2}\right) = -1 + \frac{a}{2}$$

from which we conclude that $a = -2$.

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