MA10207 Analysis 1 - Semester 1, 2021/22

Problem Sheet Week 4 - Solutions

1. Using the definition of limits, prove that

$$(a). \qquad \lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2}, \qquad (b). \qquad \lim_{n\to\infty} \frac{\cos(10^{2n^2})}{n^2} = 0.$$

[*Hint: you may use that* $-1 \leq \cos(x) \leq 1$ *for all* $x \in \mathbb{R}$.]

Answer. (a). We want to prove that

$$(\forall \varepsilon > 0) \quad (\exists N \in \mathbb{N}), \quad \forall n \geq N \quad \left|\frac{n}{2n+3} - \frac{1}{2}\right| < \varepsilon.$$

Observe that

$$\left|\frac{n}{2n+3} - \frac{1}{2}\right| = \frac{3}{4n+6}.$$

Take $\varepsilon > 0$. We have

$$\left|\frac{n}{2n+3} - \frac{1}{2}\right| < \varepsilon \quad \Leftrightarrow \quad \frac{3}{4n+6} < \varepsilon \quad \Leftrightarrow \quad n > \frac{3-6\varepsilon}{4\varepsilon}.$$

By the Archimedean Principle, there exists $N \in \mathbb{N}$ such that $N > \frac{3-6\varepsilon}{4\varepsilon}$. Hence for all $n \geq N$, $|\frac{n}{2n+3} - \frac{1}{2}| < \varepsilon$.

(b). We want to prove that

$$(\forall \varepsilon > 0) \quad (\exists N \in \mathbb{N}), \quad \forall n \geq N \quad \left|\frac{\cos(10^{2n^2})}{n^2}\right| < \varepsilon.$$

Take $\varepsilon > 0$. Observe that

$$\left|\frac{\cos(10^{2n^2})}{n^2}\right| = \frac{|\cos(10^{2n^2})|}{n^2} \leq \frac{1}{n^2}$$

Hence

$$n > \frac{1}{\sqrt{\varepsilon}} \quad \Rightarrow \quad \left|\frac{\cos(10^{2n^2})}{n^2}\right| \leq \frac{1}{n^2} < \varepsilon.$$

It is enough to choose $N \in \mathbb{N}$ with $N > \frac{1}{\sqrt{\varepsilon}}$. Such N exists by the Archimedean Principle.

2. Find the limit of the following sequences

$$(i) \quad a_n = 1 + (\frac{1}{3})^n, \qquad (ii) \qquad a_n = \frac{7^n(1-n)}{(1+n^2)9^n}.$$

Answer. (i) From the lecture notes we know that $\lim_{n\to\infty}(\frac{1}{3})^n = 0$. By the Algebra of Limits,

$$\lim_{n\to\infty} a_n = 1.$$

(ii) We simplify

$$a_n = \frac{7^n(1-n)}{(1+n^2)9^n} = (\frac{7}{9})^n \frac{(\frac{1}{n}-1)}{n(\frac{1}{n^2}+1)}$$

From the lecture notes, we have that $\lim_{n\to\infty}(\frac{7}{9})^n = 0$, $\lim_{n\to\infty}\frac{1}{n} = 0$. By the algebra of limits, we conclude that

$$\lim_{n\to\infty} \frac{7^n(1-n)}{(1+n^2)9^n} = 0.$$

3. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $\lim_{n\to\infty} a_n = L \geq 0$.
   a. Prove that $(\sqrt{a_n})_{n\in\mathbb{N}}$ converges to $\sqrt{L}$.
   b. Prove that $(\sqrt{1+a_n^2})_{n\in\mathbb{N}}$ converges to $\sqrt{1+L^2}$.

Answer. *a.* For any $x, y \geq 0$ it holds

$$|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|}.$$

This fact has been proven in the Problem Sheet Week 3.

Hence, for all n,

$$|\sqrt{a_n} - \sqrt{L}| \leq \sqrt{|a_n - L|}.$$

Since $\lim_{n\to\infty} a_n = L$, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that for any $n \geq N$

$$|a_n - L| < \varepsilon^2.$$

We have that: for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that for any $n \geq N$

$$|\sqrt{a_n} - \sqrt{L}| \leq \sqrt{\varepsilon^2} = \varepsilon.$$

This concludes the proof.

b. In Problem Sheet Week 3 we proved that, for any $x, y \in \mathbb{R}$

$$|\sqrt{1+x^2} - \sqrt{1+y^2}| \leq |x-y|.$$

Thus, for any n, we have

$$|\sqrt{1+a_n^2} - \sqrt{1+L^2}| \leq |a_n - L|.$$

Since $\lim_{n\to\infty} a_n = L$, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that for any $n \geq N$

$$|a_n - L| < \varepsilon.$$

We have that: for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that for any $n \geq N$

$$|\sqrt{1+a_n^2} - \sqrt{1+L^2}| < \varepsilon.$$

This concludes the proof.

4. Let $A \subseteq \mathbb{R}$ be a set bounded above, and $A \neq \emptyset$. Show that there exists a sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n \in A$ for any $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \sup(A)$.

Answer. Since A is bounded above, and $A \neq \emptyset$, $\sup(A)$ exists by the Completeness Axiom. By definition of $\sup(A)$, for any $n \in \mathbb{N}$ there exists $a_n \in A$ so that

$$\sup(A) - \frac{1}{n} \leq a_n.$$

Moreover, for all $n \in \mathbb{N}$, $a_n \leq \sup(A)$. Hence, for all n

$$\sup(A) - \frac{1}{n} \leq a_n \leq \sup(A).$$

We claim that $\lim_{n\to\infty} a_n = \sup(A)$. Observe that

$$0 \leq \sup(A) - a_n \leq \frac{1}{n}.$$

By the Archimedean Postulate, for any ε there exists $N \in \mathbb{N}$ so that $N > \frac{1}{\varepsilon}$. Therefore, for any $n \geq N$,

$$|a_n - \sup(A)| < \varepsilon.$$

Hence $\lim_{n\to\infty} a_n = \sup(A)$.

Homework.

1. Using the definition of limits, prove that

$$a. \quad \lim_{n\to\infty} \frac{2n+3}{3n-7} = \frac{2}{3}, \qquad b. \quad \lim_{n\to\infty} \frac{n}{3n^2+2} = 0.$$

Answer. *a.* We need to show that for any real number $\varepsilon > 0$ there exists an integer number N so that, for any $n \geq N$ then

$$|\frac{2n+3}{3n-7} - \frac{2}{3}| < \varepsilon$$

We compute

$$\frac{2n+3}{3n-7} - \frac{2}{3} = \frac{23}{3(3n-7)}.$$

Hence the inequality gets rewritten as

$$\frac{23}{3|3n-7|} < \varepsilon \Leftrightarrow |3n-7| > \frac{23}{3\varepsilon}.$$

We can now assume that $n \geq 3$, so $|3n-7| = 3n-7$. Thus

$$\frac{23}{3|3n-7|} < \varepsilon \quad \Leftrightarrow |3n-7| > \frac{23}{3\varepsilon} \quad \Leftrightarrow \quad n > \frac{21\varepsilon+23}{9\varepsilon}.$$

By the Archimedean Postulate, given any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that $N > \max(3, \frac{14\varepsilon+69}{6\varepsilon})$. This concludes the proof.

b. We want to prove that

$$(\forall \varepsilon > 0) \quad (\exists N \in \mathbb{N}), \quad \forall n \geq N \quad |\frac{n}{3n^2+2}| < \varepsilon.$$

Take $\varepsilon > 0$. Observe that

$$|\frac{n}{3n^2+2}| \leq \frac{n}{3n^2} = \frac{1}{3n}$$

Hence

$$n > \frac{1}{3\varepsilon} \quad \Rightarrow \quad |\frac{n}{3n^2+2}| \leq \frac{1}{3n} < \varepsilon.$$

It is enough to choose $N \in \mathbb{N}$ with $N > \frac{1}{3\varepsilon}$. Such N exists by the Archimedean Principle.

2. Find the limit of the following sequences

$$(i) \quad a_n = \sqrt{n^2+1} - n, \quad (ii) \quad a_n = \frac{1 - (\frac{2}{3})^n}{n^{-4} + n^{-3} - 9}.$$

[Hint for (i): calculate first $(\sqrt{n^2+1} - n)(\sqrt{n^2+1} + n)$.]

Answer. (i) We rationalise

$$a_n = \sqrt{n^2+1} - n = \frac{(\sqrt{n^2+1} - n)(\sqrt{n^2+1} + n)}{\sqrt{n^2+1} + n} = \frac{1}{\sqrt{n^2+1} + n}.$$

We factor out n in the denominator

$$a_n = \frac{1}{\sqrt{n^2+1} + n} = \frac{1}{n}\left(\frac{1}{\sqrt{1 + \frac{1}{n^2}} + 1}\right).$$

From the lecture notes, we have that $\lim_{n\to\infty} \frac{1}{n} = 0$. By the algebra of limits, $\lim_{n\to\infty} \frac{1}{n^2} = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{n} = 0$. We claim that $\lim_{n\to\infty} \sqrt{1 + \frac{1}{n^2}} = 1$. Indeed, given $\varepsilon > 0$, one has

$$|\sqrt{1 + \frac{1}{n^2}} - 1| < \varepsilon \quad \Leftrightarrow \quad \sqrt{1 + \frac{1}{n^2}} - 1 < \varepsilon \quad \Leftrightarrow \sqrt{1 + \frac{1}{n^2}} < 1 + \varepsilon \quad \Leftrightarrow \quad 1 + \frac{1}{n^2} < (1+\varepsilon)^2$$
$$\Leftrightarrow \quad n > \frac{1}{\sqrt{2\varepsilon + \varepsilon^2}}.$$

For any $\varepsilon > 0$, by the Archimedean postulate there exists $N \in \mathbb{N}$ so that $N > \frac{1}{\sqrt{2\varepsilon+\varepsilon^2}}$. Thus, for any $n \geq N$, it holds

$$|\sqrt{1 + \frac{1}{n^2}} - 1| < \varepsilon.$$

This proves that $\lim_{n\to\infty} \sqrt{1 + \frac{1}{n^2}} = 1$. By the algebra of limits, we get that

$$\lim_{n\to\infty} \sqrt{1 + \frac{1}{n^2}} + 1 = 2, \quad \lim_{n\to\infty} \left(\frac{1}{\sqrt{1 + \frac{1}{n^2}} + 1}\right) = \frac{1}{2}$$

and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}+n} = \lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{\sqrt{1+\frac{1}{n^2}}+1}\right) = 0.$$

(ii) From the lecture notes, we have that $\lim_{n\to\infty}(\frac{2}{3})^n = 0$. By the algebra of limits, we get $\lim_{n\to\infty} 1-(\frac{2}{3})^n = 1$. Besides, since $\lim_{n\to\infty} \frac{1}{n} = 0$, by the Algebra of Limits we have $\lim_{n\to\infty} n^{-4} = \lim_{n\to\infty} n^{-3} = 0$ and $\lim_{n\to\infty} n^{-4}+n^{-3}-9 = -9$.

Using again the Algebra of Limits we get

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[\frac{1-(\frac{2}{3})^n}{n^{-4}+n^{-3}-9}\right] = -\frac{1}{9}.$$

3. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers, such that $\lim_{n\to\infty} a_n = 1$. Using the definition of limits, prove that $\lim_{n\to\infty} a_n^{\frac{1}{3}} = 1$.

Answer. We want to prove that

$$(\forall\varepsilon > 0) \quad (\exists N \in \mathbb{N}) \quad \forall n \geq N \quad |a_n^{\frac{1}{3}} - 1| < \varepsilon.$$

Recall that

$$x^3 - y^3 = (x-y)(x^2+xy+y^2) \quad \forall x, y \in \mathbb{R}.$$

Hence

$$a_n - 1 = (a_n^{\frac{1}{3}} - 1)(a_n^{\frac{2}{3}} + a_n^{\frac{1}{3}} + 1).$$

Let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = 1$,

$$\exists N_1 \in \mathbb{N} \quad \forall n \geq N_1 \quad |a_n - 1| < \frac{1}{2},$$

and

$$\exists N_2 \in \mathbb{N} \quad \forall n \geq N_2 \quad |a_n - 1| < \varepsilon.$$

Observe that $|a_n - 1| < \frac{1}{2} \Leftrightarrow \frac{1}{2} < a_n < \frac{3}{2}$, thus $\forall n \geq N_1$ we have that $a_n > 0$.

Take $N = \max(N_1, N_2)$. Then $\forall n \geq N$

$$|a_n^{\frac{1}{3}} - 1| = |\frac{a_n - 1}{a_n^{\frac{2}{3}} + a_n^{\frac{1}{3}} + 1}| = \frac{|a_n - 1|}{a_n^{\frac{2}{3}} + a_n^{\frac{1}{3}} + 1} \leq |a_n - 1| < \varepsilon.$$

This concludes the proof.

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