MA10207 Analysis 1 - Semester 1, 2021/22

Problem Sheet Week 3 - Solutions

1. Let A be a non-empty bounded set in $\mathbb{R}$. Let the *diameter* of A be defined as

$$\mathrm{diam}(A) := \sup\{|x-y| : x, y \in A\}.$$

(a.) Prove that the number $\sup(A) - \inf(A)$ is an upper bound for the set $\{|x-y| : x, y \in A\}$.

(b.) Prove that $\mathrm{diam}(A) \leq \sup(A) - \inf(A)$.

(c.) Prove that $\mathrm{diam}(A) = \sup(A) - \inf(A)$.

Answer. (a.) Since A is a non-empty bounded set in $\mathbb{R}$, $\sup(A)$ and $\inf(A)$ exist by the Completeness Axiom. Let $x, y \in A$, and assume that $x \geq y$ (otherwise we exchange the role of x and y). By definition of $\sup(A)$ and $\inf(A)$, we have

$$x \leq \sup(A), \quad y \geq \inf(A).$$

Hence

$$|x-y| = x - y \leq \sup(A) - \inf(A) \quad \forall \quad x, y \in A.$$

Thus $\sup(A) - \inf(A)$ is an upper bound for the set $\{|x-y| : x, y \in A\}$.

(b.) By definition, $\mathrm{diam}(A)$ is the least upper bound of the set $\{|x-y| : x, y \in A\}$, hence

$$\mathrm{diam}(A) = \sup\{|x-y| : x, y \in A\} \leq \sup(A) - \inf(A).$$

(c.) It is enough to prove that $\mathrm{diam}(A) \geq \sup(A) - \inf(A)$.

By contradiction, assume that $\mathrm{diam}(A) < \sup(A) - \inf(A)$. Hence there exists $\varepsilon > 0$ so that

$$\mathrm{diam}(A) < \sup(A) - \inf(A) - 2\varepsilon = \sup(A) - \varepsilon - (\inf(A) + \varepsilon). \tag{1}$$

By the definitions of $\sup(A)$ and $\inf(A)$, there exist $\bar{x} \in A$ and $\bar{y} \in A$ such that

$$\sup(A) - \varepsilon < \bar{x}, \quad \inf(A) + \varepsilon > \bar{y}.$$

Substituting in (1), we get

$$\mathrm{diam}(A) < \sup(A) - \varepsilon - (\inf(A) + \varepsilon) < \bar{x} - \bar{y} \leq |\bar{x} - \bar{y}| \leq \mathrm{diam}(A),$$

which is impossible. This implies that $\mathrm{diam}(A) \geq \sup(A) - \inf(A)$.

2. Solve the inequalities

$$a. \quad \frac{1}{x} + \frac{1}{x+1} > 0; \qquad b. \quad |x-1| + |x-2| \geq 5.$$

Answer. a. We write

$$\frac{1}{x} + \frac{1}{x+1} > 0 \quad \Leftrightarrow \quad \frac{2x+1}{x(x+1)} > 0.$$

For the values $x = 0$ or $x = -1$ the inequality is meaningless so we rule these values out straight away.

(i) Consider only $x < -1$. Then $x < 0$ and $x + 1 < 0$, thus $x(x+1) > 0$ and $\frac{2x+1}{x(x+1)} > 0 \quad \Leftrightarrow \quad 2x+1 > 0 \Leftrightarrow \quad x > -\frac{1}{2}$, which is impossible for this case.

(ii) Consider only $-1 < x < 0$. Then $x(x+1) < 0$ and $\frac{2x+1}{x(x+1)} > 0 \quad \Leftrightarrow \quad 2x+1 < 0 \Leftrightarrow$ $x < -\frac{1}{2}$. So we have solutions for the x under consideration exactly when $-1 < x < -\frac{1}{2}$.

(iii) Consider only $x > 0$. Then $x(x+1) > 0$ as in (i) and $\frac{2x+1}{x(x+1)} > 0 \quad \Leftrightarrow \quad x > -\frac{1}{2}$. So the solutions for those x under consideration are exactly $x > 0$.

It follows that the solution set is exactly those x such that either $-1 < x < -\frac{1}{2}$ or $x > 0$.

b. Consider only $x \leq 1$. In this case

$$|x-1| = 1 - x, \quad |x-2| = 2 - x$$

hence the inequality becomes $1 - x + 2 - x \geq 5 \Leftrightarrow \quad -2x \geq 2 \Leftrightarrow \quad x \quad \leq -1$. So we have solutions for the x under consideration exactly when $x \leq -1$.

Consider now only $1 < x \leq 2$. In this case

$$|x-1| = x - 1, \quad |x-2| = 2 - x$$

hence the inequality becomes $x - 1 + 2 - x \geq 5 \Leftrightarrow 1 \geq 5$, which is impossible. So we have no solutions for the x under consideration.

Consider now only $x > 2$. In this case

$$|x-1| = x - 1, \quad |x-2| = x - 2$$

hence the inequality becomes $x - 1 + x - 2 \geq 5 \Leftrightarrow 2x \geq 8 \Leftrightarrow x \geq 4$. So we have solutions for the x under consideration exactly when $x \geq 4$.

It follows that the solution set is exactly $(-\infty, -1] \cup [4, \infty)$.

3. a. Show that

$$2xy \le x^2 + y^2, \qquad \forall \quad x, y \in \mathbb{R}$$

and that equality holds only if $x = y$.

b. Show that

$$\sqrt{\frac{x}{2}} + \sqrt{\frac{y}{2}} \le \sqrt{x+y} \le \sqrt{x} + \sqrt{y}, \qquad \forall \quad x, y > 0.$$

c. Prove that

$$\left| \sqrt{1+x^2} - \sqrt{1+y^2} \right| \le |x - y|, \qquad \forall \quad x, y \in \mathbb{R}.$$

Answer. Proof of a. For any $x, y \in \mathbb{R}$, $(x-y)^2 \ge 0$, and equality holds only if $x = y$. Thus

$$x^2 - 2xy + y^2 \ge 0, \quad \forall x, y.$$

Rearranging the terms we get

$$2xy \le x^2 + y^2,$$

and equality holds only if $x = y$.

Proof of b. We first prove $\sqrt{x+y} \le \sqrt{x} + \sqrt{y}$. We have

$$(\sqrt{x} + \sqrt{y})^2 = x + y + 2\sqrt{xy} \ge x + y.$$

Taking the square root in both sides we obtain

$$\sqrt{x+y} \le \sqrt{x} + \sqrt{y}.$$

We next prove $\sqrt{\frac{x}{2}} + \sqrt{\frac{y}{2}} \le \sqrt{x+y}$. We have

$$\left(\sqrt{\frac{x}{2}} + \sqrt{\frac{y}{2}}\right)^2 = \frac{x}{2} + \frac{y}{2} + 2\sqrt{\frac{x}{2}}\sqrt{\frac{y}{2}} \underbrace{\le}_{by\, part\, a.} \frac{x}{2} + \frac{y}{2} + \frac{x}{2} + \frac{y}{2} = x + y.$$

Taking the square root of both sides, we get the wanted inequality.

Proof of c. Let us assume first that $x \ge y$. Thus $\sqrt{1+x^2} - \sqrt{1+y^2} > 0$ and the inequality to prove becomes

$$\sqrt{1+x^2} - \sqrt{1+y^2} \le x - y.$$

We have

$$0 \le \left(\sqrt{1+x^2} - \sqrt{1+y^2}\right)^2 \le 1 + x^2 + 1 + y^2 - 2\sqrt{1+x^2}\sqrt{1+y^2}$$
$$= 2 + x^2 + y^2 - 2\sqrt{1+x^2+y^2+x^2y^2}.$$

The inequality in part a. says that

$$2xy \le x^2 + y^2 \Rightarrow -2\sqrt{1+x^2+y^2+x^2y^2} \le -2\sqrt{1+2xy+x^2y^2} = -2(1+xy),$$

hence

$$0 \le \left(\sqrt{1+x^2} - \sqrt{1+y^2}\right)^2 \le 1 + x^2 + 1 + y^2 - 2\sqrt{1+x^2}\sqrt{1+y^2}$$
$$= 2 + x^2 + y^2 - 2\sqrt{1+x^2+y^2+x^2y^2}$$
$$\le 2 + x^2 + y^2 - 2(1+xy) = (x-y)^2.$$

Thus we conclude that $\sqrt{1+x^2} - \sqrt{1+y^2} \le x - y$. The case $x \le y$ can be treated similarly.

Homework.

1. Solve the inequalities

$$a. \quad -1 \le \frac{x-1}{2x+1} \le 1, \quad b. \quad |x-1| - 3 > -1.$$

Answer. a. We consider the two inequalities separately. Therefore, we write the set of solutions in the form $A \cap B$, where $A = \{x \in \mathbb{R} \setminus \{-\frac{1}{2}\} : -1 \le \frac{x-1}{2x+1}\}$ and $B = \{x \in \mathbb{R} \setminus \{-\frac{1}{2}\} : 1 \ge \frac{x-1}{2x+1}\}$.

Find A. We have

$$\frac{x-1}{2x+1} \ge -1 \quad \Leftrightarrow \quad \frac{x-1}{2x+1} + 1 \ge 0 \quad \Leftrightarrow \quad \frac{x-1+2x+1}{2x+1} \ge 0.$$

If $x > -\frac{1}{2}$, this is equivalent to

$$3x \ge 0 \quad \Leftrightarrow \quad x \ge 0.$$

If $x < -\frac{1}{2}$,

$$3x \le 0 \quad \Leftrightarrow \quad x \le 0.$$

We thus conclude that $A = (-\infty, -\frac{1}{2}) \cup [0, \infty)$.

Find B. We have

$$\frac{x-1}{2x+1} \le 1 \quad \Leftrightarrow \quad \frac{x-1}{2x+1} - 1 \le 0 \quad \Leftrightarrow \quad \frac{x-1-2x-1}{2x+1} \le 0.$$

If $x > -\frac{1}{2}$, this is equivalent to

$$-x - 2 \leq 0 \quad \Leftrightarrow \quad x \geq -2.$$

If $x < -\frac{1}{2}$,

$$-x - 2 \geq 0 \quad \Leftrightarrow \quad x \leq -2.$$

Thus $B = (-\infty, -2] \cup (-\frac{1}{2}, \infty)$.

We conclude that $A \cap B = (-\infty, -2] \cup [0, \infty)$.

b. We solve $|x - 1| > 2$.

If $x \geq 1$, the inequality becomes $x - 1 > 2$, $x > 3$. Hence in this region the inequality is satisfied for $x > 3$.

If $x < 1$, the inequality becomes $-x + 1 > 2$, $x < -1$. Hence in this region the inequality is satisfied for $x < -1$.

We conclude that the solution set is $(-\infty, -1) \cup (3, \infty)$.

2. Let $A \subseteq B$ be non-empty subsets of $\mathbb{R}$.
   a. Prove that if B has a supremum, then A has a supremum and $\sup(A) \leq \sup(B)$.
   b. Prove that if B has an infimum, then A has an infimum and $\inf(B) \leq \inf(A)$.

Answer.

a. If $s = \sup(B)$ then for all $b \in B$, $b \leq s$. Since $A \subseteq B$, for every $a \in A$, we have that $a \in B$ and then $a \leq s$. Thus s is an upper bound for A. By the Completeness Axiom, $\sup(A)$ exists. Moreover, $\sup(A) \leq s$, by the least upper bound property.

b. If $\ell = \inf(B)$ then for all $b \in B$, $\ell \leq b$. Since $A \subseteq B$, for every $a \in A$, $a \in B$ and hence $\ell \leq a$. Thus ℓ is a lower bound for A. By the Completeness Axiom, $\inf(A)$ exists. Moreover, $\inf(A) \geq \ell$ by the definition of infimum.

3. Prove that

$$|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|}$$

for all $x, y \geq 0$.

Answer. Let us assume that $x \geq y$. Then the inequality to prove becomes

$$\sqrt{x} - \sqrt{y} \leq \sqrt{x - y}.$$

We have

$$(\sqrt{x} - \sqrt{y})^2 = x + y - 2\sqrt{x}\sqrt{y}$$

Since $\sqrt{x} \geq \sqrt{y}$, we have $-2\sqrt{x}\sqrt{y} \leq -2y$. We insert this information in the inequality above and obtain

$$(\sqrt{x} - \sqrt{y})^2 = x + y - 2\sqrt{x}\sqrt{y} \leq x - y \;\Rightarrow\quad \sqrt{x} - \sqrt{y} \leq \sqrt{x - y}.$$

The case $x \leq y$ can be treated in a similar way.

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