clarifying

Q2 (c) &

Q4

4. Prove that
$$\left(\sum_{n=0}^{\infty} \frac{7^n}{n!}\right) \times \left(\sum_{n=0}^{\infty} \frac{(-1)^n(7)^n}{n!}\right) = 1.$$

Proof; Cauchy product of series is:

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} \underbrace{\frac{7^n}{n!}}_{= a_n}\right)\left(\sum_{n=0}^{\infty} \underbrace{\frac{(-1)^n(7)^n}{n!}}_{= b_n}\right)$$

we know $c_n = \sum_{k=0}^{n} a_k b_{n-k}$

Thus:

$$c_n = \sum_{k=0}^{n} \left(\frac{7^k}{k!}\right)\left(\frac{(-1)^{n-k} 7^{n-k}}{(n-k)!}\right)$$

using Binomial formula for $n \geq 1$

$$= \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} 7^k (-7)^{n-k}$$

$$= \frac{1}{n!} (7 + (-7))^n = 0$$

Also, if $n=0$: $c_n = 1$

$$\Rightarrow \sum_{n=0}^{\infty} c_n = 1 \quad \blacksquare$$

2. For each of the following power series, find the interval of convergence and the radius of convergence:

$$a. \sum_{n=1}^{\infty}(-1)^n n^2 x^n \quad , \quad b. \sum_{n=1}^{\infty}\frac{2^n}{n^2}(x-3)^n, \quad c. \sum_{n=1}^{\infty}\frac{n^3}{3^n}(x+1)^n$$

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c. $\sum_{n=1}^{\infty} \frac{n^3}{3^n}(x+1)^n$

$Z := x+1$ and consider

$$\sum_{n=1}^{\infty} \frac{n^3}{3^n} Z^n$$

$$R = \lim_{n\to\infty} \left| \frac{n^3\, 3^{n+1}}{(n+1)^3\, 3^n} \right| = 3$$

So $\forall Z \in (-3, 3)$, $\sum_{n=1}^{\infty} \frac{n^3}{3^n} Z^n$ Converges.

If $Z = -3$: our series become $\sum_{n=1}^{\infty} n^3$

since $\lim_{n\to\infty} n^3 \neq 0$

similarly we can show it for $Z = 3$

Thus the Convergence interval is $(-3, 3)$

so $\forall x \in (-4, 2)$ our original series $\sum_{n=1}^{\infty} \frac{n^3}{3^n}(x+1)^n$ Converges.